Sign changing solutions of Poisson's equation

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Abstract

Let Ω be an open, possibly unbounded, set in Euclidean space $\mathbb{R}^m$ with boundary $\partial\Omega$, let A be a measurable subset of Ω with measure $|A|$, and let $\gamma \in (0,1)$. We investigate whether the solution $v_{\Omega,A,\gamma}$ of $-\Delta v = \gamma \mathbf{1}_{\Omega\setminus A} - (1-\gamma)\mathbf{1}_A$ with $v = 0$ on $\partial\Omega$ changes sign. Bounds are obtained for $|A|$ in terms of geometric characteristics of Ω (bottom of the spectrum of the Dirichlet Laplacian, torsion, measure, or R-smoothness of the boundary) such that $\text{essinf} v_{\Omega,A,\gamma} \ge 0$. We show that $\text{essinf} v_{\Omega,A,\gamma} < 0$ for any measurable set A, provided $|A| > \gamma|\Omega|$. This value is sharp. We also study the shape optimisation problem of the optimal location of A (with prescribed measure) which minimises the essential infimum of $v_{\Omega,A,\gamma}$. Surprisingly, if Ω is a ball, a symmetry breaking phenomenon occurs.

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1 Introduction

Let Ω be an open, possibly unbounded, set in Euclidean space $\mathbb{R}^m$ with boundary $\partial\Omega$, and with, possibly infinite, measure $|\Omega|$. It is well-known [3] that if the bottom of the Dirichlet Laplacian defined by

$$\lambda(\Omega) = \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |D\varphi|^2}{\displaystyle\int_\Omega \varphi^2},$$

is bounded away from 0, then

$$-\Delta v = 1,\ v = 0 \text{ on } \partial\Omega, \tag{1}$$

has a unique weak solution denoted by v_Ω, which is non-negative, and which satisfies,

$$\lambda(\Omega)^{-1} \le \|v_\Omega\|_{L^\infty(\Omega)} \le (4 + 3m\log 2)\lambda(\Omega)^{-1}. \tag{2}$$

The m-dependent constant in the right-hand side of (2) has been improved in [14], and subsequently in [25].

If $|\Omega| < \infty$ then, by the Faber-Krahn inequality, $\lambda(\Omega) > 0$, and by (2), $v_\Omega \in H_0^1(\Omega)$, and $v_\Omega \in L^1(\Omega)$. For an arbitrary open set Ω we define the torsion, or torsional rigidity, by

$$T(\Omega) = \int_\Omega v_\Omega.$$

Note that, under the assumption $\lambda(\Omega) > 0$, by (2) the solution of an equation like in (1) with a right-hand side $f \in L^\infty(\Omega)$ can be defined by approximation on balls for the positive and negative parts of f.

For a measurable subset $A \subset \Omega$, with $\lambda(\Omega) > 0$, and $0 < \gamma < 1$, we denote by $v_{\Omega,A,\gamma}$ the solution of

$$-\Delta v = \gamma \mathbf{1}_{\Omega\setminus A} - (1-\gamma)\mathbf{1}_A,\ v = 0 \text{ on } \partial\Omega. \tag{3}$$

These hypotheses on A, Ω and γ will not be repeated in the statements of all lemmas and theorems below.

This paper investigates whether the solution of (3) satisfies essinf $v_{\Omega,A,\gamma} < 0$. Whether this holds depends on the geometry of Ω, and on the size and the location of the set $A \subset \Omega$. This question shows up in a variety of situations. We refer, for instance, to [18], where v is a scalar potential and the right-hand side stands for a magnetic field which changes sign. The influence of the magnetic field on the asymptotic behaviour of the bottom of the spectrum of the Pauli operator is effective provided that the scalar potential has constant sign, i.e. essinf $v_{\Omega,A,\gamma} = 0$. In fluid mechanics, the function v can be interpreted as a vorticity stream function, for a vorticity taking the values γ and $-(1-\gamma)$. If $v_{\Omega,A,\gamma}$ changes sign then there exist at least two stagnation points. More situations where the sign question of the state function is put in relationship with sign changing data can be found in [8], [12], [22] and, in some biological models, [19].

Definition 1. *For $\gamma \in (0,1)$, $\Omega \subset \mathbb{R}^m$, with $\lambda(\Omega) > 0$,*

$$\mathfrak{C}_-(\Omega,\gamma) = \sup\{c \geq 0 : \forall\, A \subset \Omega, A \,\text{measurable}, |A| \leq c, \text{essinf}\, v_{\Omega,A,\gamma} \geq 0\},$$

$$\mathfrak{C}_+(\Omega,\gamma) = \inf\{c \geq 0 : \forall\, A \subset \Omega, A \,\text{measurable}, |A| > c, \text{essinf}\, v_{\Omega,A,\gamma} < 0\}.$$

It follows immediately from the definition that for a homothety $t\Omega$, $t > 0$ of Ω we have the scaling relations

$$\mathfrak{C}_-(t\Omega,\gamma) = t^m \mathfrak{C}_-(\Omega,\gamma), \tag{4}$$

and

$$\mathfrak{C}_+(t\Omega,\gamma) = t^m \mathfrak{C}_+(\Omega,\gamma).$$

Furthermore if Ω_1, Ω_2 are disjoint open sets, then

$$\mathfrak{C}_-(\Omega_1 \cup \Omega_2,\gamma) = \min\{\mathfrak{C}_-(\Omega_1,\gamma), \mathfrak{C}_-(\Omega_2,\gamma)\},$$

$$\mathfrak{C}_+(\Omega_1 \cup \Omega_2,\gamma) = \mathfrak{C}_+(\Omega_1,\gamma) + \mathfrak{C}_+(\Omega_2,\gamma).$$

This paper concerns the analysis of these quantities and their dependence on Ω. It turns out that $\mathfrak{C}_+(\Omega,\gamma) = \gamma|\Omega|$ for arbitrary open sets Ω with finite measure. On the contrary, $\mathfrak{C}_-(\Omega,\gamma)$ is very sensitive to the geometry. We find its main properties, give basic estimates, establish isoperimetric and isotorsional inequalities, and we discuss the shape optimisation problem related to the optimal location of the set A in order to minimise the essential infimum.

Theorem 1. *For every non-empty open set $\Omega \subseteq \mathbb{R}^m$ of finite measure we have*

$$\mathfrak{C}_+(\Omega,\gamma) = \gamma|\Omega|.$$

Below we show that, in general, we have to assume some regularity of Ω in order to have $\mathfrak{C}_-(\Omega,\gamma) > 0$. For instance, if $\Omega = \cup_{j\in\mathbb{N}} C_j$ is a set of finite measure, where the sets $C_j, j \in \mathbb{N}$ are non-empty, open, disjoint, then $\mathfrak{C}_-(\Omega,\gamma) = 0$. Indeed, if we let $A = C_j$ then essinf $v_{\Omega,A,\gamma} \leq (\gamma-1)$esssup $v_{C_j} < 0$. Consequently, $\mathfrak{C}_-(\Omega,\gamma) \leq |C_j|$ for every j, so $\mathfrak{C}_-(\Omega,\gamma) = 0$.

Theorem 2. *If $\Omega \subset \mathbb{R}^2$ is any open triangle, then $\mathfrak{C}_-(\Omega,\gamma) = 0$.*

In Theorem 3 below we show that if Ω is bounded, and $\partial\Omega$ is of class C^2 then $\mathfrak{C}_-(\Omega) > 0$. In order to quantify this assertion we introduce some notation. For a non-empty open set Ω we denote by $\text{diam}(\Omega) = \sup\{|x-y| : x \in \Omega, y \in \Omega\}$. We denote the complement $\mathbb{R}^m \setminus E$ of E by E^c, and the closure of E by $\overline{E}$. Furthermore, $B_r(x) := \{y \in \mathbb{R}^m : |x-y| < r\}$ denotes the open ball centred at x of radius r. If $x = 0$, we simply write B_r. We set $\omega_m = |B_1|$. For $x \in \Omega$ we let $\bar{x} \in \partial\Omega$ be a point such that $|x - \bar{x}| = \min\{|x-z| : z \in \partial\Omega\}$. We recall the following from [2, p.280].

Definition 2. *An open set $\Omega \subset \mathbb{R}^m$, $m \geq 2$, has R-smooth boundary if at any point $x_0 \in \partial\Omega$, there are two open balls $B_R(x_1)$, $B_R(x_2)$ such that $B_R(x_1) \subset \Omega$, $B_R(x_2) \subset \mathbb{R}^m \setminus \bar{\Omega}$ and $\bar{B}_R(x_1) \cap \bar{B}_R(x_2) = \{x_0\}$.*

We also recall that a bounded Ω with C^2 boundary $\partial\Omega$ is R-smooth for some $R > 0$.

Theorem 3. *If Ω is an open, bounded set in $\mathbb{R}^m$ with a C^2 and R-smooth boundary, then*

$$\mathfrak{C}_-(\Omega,\gamma) \ge \mathfrak{C}_-(B_R,\gamma).$$

Furthermore

$$\mathfrak{C}_-(B_R,\gamma) \ge \left(\frac{\gamma}{4m}\right)^m \omega_m R^m,\ m = 2,3,..., \tag{5}$$

$$\mathfrak{C}_-(B_R,\gamma) \le \gamma^{m/2}\omega_m R^m,\ m \ge 3, \tag{6}$$

and

$$\mathfrak{C}_-(B_R,\gamma) \le \left(1+\log\left(\frac{1}{\gamma}\right)\right)^{-1}\gamma\pi R^2,\ m = 2. \tag{7}$$

The following inequality gives an upper bound for $\mathfrak{C}_-(\Omega,\gamma)$ in terms of $\lambda(\Omega)$.

Theorem 4. *For every open set $\Omega \subset \mathbb{R}^m$ with $\lambda(\Omega) > 0$,*

$$\mathfrak{C}_-(\Omega,\gamma) \le C_1(m)\left(\frac{\gamma}{1-\gamma}\right)^{m/2}\lambda(\Omega)^{-m/2},$$

where

$$C_1(m) = \omega_m^{(m+2)/2}2^{5m^2/12}3^{m(m+2)/4}e^{2^{1/m}\lambda(B_1)^{1/2}/24}\left(\frac{12m(m+2)}{eC_2(m)^{1/2}}\right)^{m(m+2)/2}, \tag{8}$$

and where $C_2(m)$ is the constant in the Kohler-Jobin inequality (37) *below.*

This implies that if Ω is an open set with $T(\Omega) < \infty$, then

$$\mathfrak{C}_-(\Omega,\gamma) \le C_1(m)C_2(m)^{-m/2}\left(\frac{\gamma}{1-\gamma}\right)^{m/2}T(\Omega)^{m/(m+2)}. \tag{9}$$

The optimal coefficient of $T(\Omega)^{m/(m+2)}$ in (9) is not known. However, the Kohler-Jobin inequality suggests to prove (or disprove) optimality for balls.

Theorem 5. *There exists $C_3(m) < \infty$ such that for every open, connected set $\Omega \subset \mathbb{R}^m$ with $T(\Omega) < \infty$,*

$$\mathfrak{C}_-(\Omega,\gamma) \le C_3(m)\max\left\{\left(\frac{\gamma}{1-\gamma}\right)^{\frac{m}{2}},\left(\frac{\gamma}{1-\gamma}\right)^{\frac{m(m+2)}{2(m+1)}}\right\}\left(\frac{T(\Omega)}{\mathrm{diam}(\Omega)}\right)^{m/(m+1)}. \tag{10}$$

In particular, if Ω is unbounded, then $\mathfrak{C}_-(\Omega,\gamma) = 0$. The value of $C_3(m)$ can be read-off from the proof in Section 6.

We see from Theorems 1 and 3 that $\mathfrak{C}_-(B_R,\gamma) < \mathfrak{C}_+(B_R,\gamma)$. The isoperimetric inequality below generalises this to arbitrary open sets with finite measure.

Theorem 6.

$$\sup\left\{\frac{\mathfrak{C}_-(\Omega,\gamma)}{|\Omega|} : \Omega \subseteq \mathbb{R}^m,\ \Omega \text{ open},\ \ 0 < |\Omega| < \infty\right\} < \gamma. \tag{11}$$

The theorem above implies that $\mathfrak{C}_-(\Omega,\gamma) < C(m,\gamma)|\Omega|$ for every open set of finite measure, with $C(m,\gamma)\gamma$. The proof of Theorem 6 relies on the relaxation of the shape optimisation problem (11) to the larger class of quasi open sets. We shall prove that the supremum is attained at some quasi open set Ω^* for which $\mathfrak{C}_-(\Omega^*,\gamma) < \gamma|\Omega^*|$.

The optimal value $C(m,\gamma) = \frac{\mathfrak{C}_-(\Omega^*,\gamma)}{|\Omega^*|}$ is not known, nor whether Ω^* is open. The symmetry breaking phenomenon for balls stated in Theorem 7 below does not support the ball to be a maximiser.

Given a constant $c \in (\mathfrak{C}_-(\Omega,\gamma), |\Omega|)$, there exists at least one set $A \subseteq \Omega$, $|A| = c$ such that essinf $v_{\Omega,A,\gamma} < 0$. A natural question is to find the best location of the set A of measure c, which minimises essinf $v_{\Omega,A,\gamma}$. This question is of particular interest for values of c close to $\mathfrak{C}_-(\Omega,\gamma)$, as this gives information on where the geometry of Ω is most sensitive to negative values. We prove the following shape optimisation result for the optimal location.

Theorem 7. *Let $\gamma \in (0,1)$ and let $\Omega \subset \mathbb{R}^m$ be an open, bounded and connected set with a smooth boundary $\partial\Omega$. For every $c \in (\mathfrak{C}_-(\Omega,\gamma), |\Omega|)$, the shape optimisation problem*

$$\min\{\text{essinf } v_{\Omega,A,\gamma} : A \subset \Omega, |A| = c\}, \tag{12}$$

has a solution. Moreover, if Ω is a ball B then, depending on the value of c, the optimal locations may be radial or not.

The existence of an optimal set relies partly on a concavity property of the shape functional $A \mapsto$ essinf $v_{\Omega,A,\gamma}$. We point out that the proof relies on both the concavity, and the analysis of optimality conditions in relationship with the PDE (see [10]). If Ω is a ball B and c is close to $|B|$, then the optimal location is a ball. If c is close to $\mathfrak{C}_-(B,\gamma)$ then the optimal location is no longer radial. This symmetry breaking phenomenon occurs at a value $c \in (\mathfrak{C}_-(B,\gamma), \gamma|B|)$, and is supported by analytical, and numerical computations.

Theorem 7 can be interpreted both as a (rather non-standard) shape optimisation problem or as an optimisation problem in a prescribed class of rearrangements, see e.g. [1]. We also refer to the paper of Burton and Toland [9] for models of steady waves with vorticity, where the distribution of the vorticity is prescribed, but we point out that our problem is essentially of different nature since the functional to be minimised is not an energy of the problem.

The proofs of Theorems 1, 2, 3, 4, 5, 6 and 7 are deferred to Sections 2, 3, 4, 5, 6, 7, and 8 below.

2 Proof of Theorem 1

In order to simplify notation, throughout the paper, if Ω is an open set and $A \subset \mathbb{R}^m$ is measurable, not necessarily contained in Ω, by $v_{\Omega,A,\gamma}$ we mean $v_{\Omega,\Omega\cap A,\gamma}$.

Proof. First assume that $\Omega \subset \mathbb{R}^m$ is an open set with finite measure. Assume $A \subset \Omega$ is a measurable set such that $v_{\Omega,A,\gamma} \geq 0$. In a first step, we shall prove that $|A| \leq \gamma|\Omega|$. As a consequence, $\mathfrak{C}_+(\Omega,\gamma) \leq \gamma|\Omega|$.

Indeed, since $v_{\Omega,A,\gamma} \geq 0$, one can use Talenti's theorem (see for instance [21, Theorem 3.1.1]) in the following way. We denote by v^* the Schwarz rearrangement of $v_{\Omega,A,\gamma}$, and by f^* the rearrangement of $\gamma 1_{\Omega\setminus A} - (1-\gamma)1_A$. There exist two positive values $0 < r_1 < r_2$ such that $f^* = \gamma 1_{B_{r_1}} - (1-\gamma)1_{B_{r_2}\setminus B_{r_1}}$, where r_1 is such that $|B_{r_1}| = |\Omega \setminus A|$ and $|B_{r_2}| = |\Omega|$. By Talenti's theorem, we get

$$0 \leq v^* \leq v_{B_{r_2},B^c_{r_1},\gamma}.$$

By elementary computations, one gets the expression for $v_{B_{r_2},B^c_{r_1},\gamma}$. Indeed, the solution $v_{B_{r_2},B^c_{r_1},\gamma}$ is radially symmetric and satisfies the equation

$$-v'' - \frac{m-1}{r}v' = \gamma 1_{[0,r_1]} - (1-\gamma)1_{[r_1,r_2]},$$

with initial condition $v'(0) = 0$, and $v(r_2) = 0$. Moreover, the solution is $C^{1,\alpha}$ regular, for some $\alpha > 0$.

We integrate separately on $[0, r_1]$, and on $[r_1, r_2]$, and write the equality of the left- and right-derivatives in r_1, namely $v'_-(r_1) = v'_+(r_1)$. Hence, we get

$$-\gamma\frac{r_1}{m} = \frac{r_2^{m-1}}{r_1^{m-1}}v'(r_2) - (1-\gamma)\frac{r_2^m}{mr_1^{m-1}} + (1-\gamma)\frac{r_1}{m}.$$

In general, from the positivity of $v_{B_{r_2},B^c_{r_1},\gamma}$ one gets that $v'(r_2) \leq 0$. Hence,

$$(1-\gamma)\frac{r_2^m}{mr_1^{m-1}} \leq \frac{r_1}{m},$$

which gives $r_1 \geq (1-\gamma)^{\frac{1}{m}} r_2$, or $|B_{r_1}| \geq (1-\gamma)|B_{r_2}|$. Finally, one gets that $|A| \leq \gamma|\Omega|$. Hence $\mathfrak{C}_+(\Omega,\gamma) \leq \gamma|\Omega|$.

As a byproduct of the computation, we observe that the constant γ in Theorem 1 is sharp, and that equality holds for the ball. As soon as, $r_1 < (1-\gamma)^{\frac{1}{m}} r_2$, one gets that $v'(r_2) > 0$. This means that as $v(r_2) = 0$ the solution is not positive near the boundary of the ball.

In order to prove the converse inequality, let us prove that for every $\varepsilon > 0$, there exists a set $A \subset \Omega$ of measure $\gamma|\Omega| - \varepsilon$ such that $v_{\Omega,A,\gamma} \geq 0$. This will imply that $\mathfrak{C}_+(\Omega,\gamma) \geq \gamma|\Omega|$.

The construction is based on the following observation. There exists a finite family of mutually disjoint balls $\cup_{i=1}^k B_i$ contained in Ω such that

$$|\Omega \setminus \cup_{i=1}^k B_i| < \varepsilon.$$

In every ball, we display the set A_i of measure $\gamma|B_i|$ in an annulus centred at the center of B_i and having ∂B_i as external boundary. Hence $v_{B_i,A_i,\gamma} \geq 0$. Moreover, since the sets B_i are mutually disjoint we get that

$$v_{\cup_i B_i,\cup_i A_i,\gamma} \geq 0.$$

We have the following.

Lemma 8. *Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^m$ be open sets with finite measure, $f \in L^2(\Omega_2)$, and let u_1, u_2 weak solutions of*

$$-\Delta u_i = f \text{ on } \Omega_i, u \in H^1_0(\Omega_i), i = 1, 2.$$

If $u_1 \geq 0$ on Ω_1 and $f \geq 0$ on $\Omega_2 \setminus \Omega_1$ then $u_2 \geq 0$ on Ω_2.

Proof. As a consequence of the hypotheses, we get

$$-\Delta u_1 \le f \text{ in } \mathcal{D}'(\Omega_2).$$

Hence, by the maximum principle

$$0 \le u_1 \le u_2 \text{ on } \Omega_2.$$

□

A direct consequence of Lemma 8 is that if $\Omega_1 \subseteq \Omega_2$ then $\mathfrak{C}_+(\Omega_1, \gamma) \le \mathfrak{C}_+(\Omega, \gamma)$. Indeed, for every measurable set $A \subseteq \Omega_1$ such that $v_{\Omega_1,A,\gamma} \ge 0$ we get $v_{\Omega_2,A,\gamma} \ge 0$.

Coming back to the proof of Theorem 1, using the additivity and monotonicity property of $\mathfrak{C}_+$ we get that

$$\mathfrak{C}_+(\Omega, \gamma) \ge \gamma |\cup_i B_i| \ge \gamma|\Omega| - \gamma\varepsilon.$$

The theorem follows by letting $\varepsilon \to 0$.

□

3 Proof of Theorem 2

We first introduce some basic notation and properties. For a non-empty open set $\Omega \subset \mathbb{R}^m$ we denote by $G_\Omega(x, y)$, $x \in \Omega$, $y \in \Omega$, $x \ne y$, the kernel of the resolvent of the Dirichlet Laplacian acting in $L^2(\Omega)$. This function exists and is well defined for all $x \ne y$, provided $m \ge 3$. It also exists for $m = 2$ for example under the hypothesis that the torsion function v_Ω defined by approximation on balls, is locally finite. The resolvent kernel is non-negative, symmetric in x and y, and is monotone increasing in Ω. That is, if $\Omega_1 \subset \Omega_2$ then,

$$0 \le G_{\Omega_1}(x, y) \le G_{\Omega_2}(x, y),\ x \in \Omega_1,\ y \in \Omega_1, x \ne y. \tag{13}$$

If v_Ω is locally finite then,

$$v_\Omega(x) = \int_\Omega dy\, G_\Omega(x, y).$$

The monotonicity in (13) implies that both the torsion function v_Ω, and torsion $T(\Omega)$ are monotone increasing in Ω.

We have also that

$$\begin{aligned} v_{\Omega,A,\gamma}(x) &= \int_\Omega dy\, G_\Omega(x, y)\big(\gamma 1_{\Omega\setminus A}(y) - (1-\gamma)1_A(y)\big) \\ &= \int_\Omega dy\, G_\Omega(x, y)\big(\gamma 1_\Omega(y) - 1_A(y)\big) \\ &= \gamma v_\Omega(x) - \int_A dy\, G_\Omega(x, y). \end{aligned} \tag{14}$$

Formula (14) implies that

$$-(1-\gamma)v_\Omega \le v_{\Omega,A,\gamma} \le \gamma v_\Omega.$$

Proof of Theorem 2. Let $\Omega = \Delta OAB$ be a triangle, with $\alpha := \angle BOA \leq \frac{\pi}{3}$ at the origin, and oriented such that the positive x-axis is the bisectrix of that angle. Let W_α be the infinite wedge with vertex at O, and edges at angles $\pm\frac{1}{2}\alpha$ with the positive x-axis, which contain the two sides OA and OB of Ω. Let $W_{\alpha,c}$ be the radial sector with area c and edges at angles $\pm\frac{1}{2}\alpha$. Then $W_{\alpha,c} \subset \Omega$ for all c sufficiently small. We have by monotonicity that

$$\begin{aligned} v_{\Omega,W_{\alpha,c},\gamma}(x) &= \int_\Omega dy\, G_\Omega(x,y)\big(\gamma\mathbf{1}_\Omega - \mathbf{1}_{W_{\alpha,c}}\big)(y) \\ &\leq \gamma \int_{W_\alpha} dy\, G_{W_\alpha}(x,y) - \int_{W_{\alpha,c}} dy\, G_{W_{\alpha,c}}(x,y) \\ &= \gamma v_{W_\alpha}(x) - v_{W_{\alpha,c}}(x). \end{aligned} \tag{15}$$

In Cartesian coordinates $x = (x_1, x_2)$ we have that

$$v_{W_\alpha}(x_1,x_2) = \frac{x_2^2 - s^2 x_1^2}{2(s^2-1)}, \tag{16}$$

where $s = \tan(\alpha/2)$. In polar coordinates $x = (r;\theta)$ we have by p.279 in [23] for the sector with radius $a = (2c/\alpha)^{1/2}$,

$$\begin{aligned} v_{W_{\alpha,c}}(r;\theta) =& \frac{r^2}{4}\left(\frac{\cos(2\theta)}{\cos\alpha} - 1\right) \\ &+ \frac{4a^2\alpha^2}{\pi^3} \sum_{n=1,3,5,\ldots} \frac{(-1)^{(n+1)/2}(r/a)^{n\pi/\alpha}\cos(n\pi\theta/\alpha)}{n\left(n+\frac{2\alpha}{\pi}\right)\left(n-\frac{2\alpha}{\pi}\right)}. \end{aligned} \tag{17}$$

We observe that for $\theta = 0$ the terms in the series in the right-hand side of (17) are alternating and decreasing in absolute value. Hence

$$v_{W_{\alpha,c}}(r;0) = \frac{r^2}{4}\left(\frac{1}{\cos\alpha} - 1\right) - \frac{4a^2\alpha^2}{\pi^3}\left(\frac{r}{a}\right)^{\pi/\alpha}\left(1 - \frac{4\alpha^2}{\pi^2}\right)^{-1}.$$

By (16) $v_{W_\alpha}(x_1,0) = \frac{s^2x_1^2}{2(1-s^2)}$, and so in polar coordinates,

$$v_{W_\alpha}(r;0) = \frac{r^2}{4}\left(\frac{1}{\cos\alpha} - 1\right). \tag{18}$$

By (15)-(18) we have

$$v_{\Omega,W_{\alpha,c},\gamma}(x_1,0) \leq (\gamma-1)\frac{s^2x_1^2}{2(1-s^2)} + O\big(x_1^{\pi/\alpha}\big),\ x_1 \downarrow 0,$$

which is negative for all x_1 sufficiently small.

We see from the proof above that we could have chosen any angle of the triangle provided that angle is strictly less than $\pi/2$. The proof above also shows that the infinite wedge $W_\alpha, \alpha < \pi/2$ with radial sector $W_{\alpha,c}, c > 0$ has a sign changing solution $v_{W_\alpha,W_{\alpha,c},\gamma}$. □

4 Proof of Theorem 3

Proof of Theorem 3. Let us start by observing that the following covering property holds: for every $x \in \Omega$, there exists a ball B of radius R such that $x \in B \subset \Omega$. Indeed, let $\bar{x} \in \partial\Omega$ be a point which realises the distance to the boundary. Since the boundary of Ω is of class C^2, then $x - \bar{x}$ is normal to the boundary $\partial\Omega$ at $\bar{x}$. If $|x - \bar{x}| \geq R$, then $B_R(x) \subset \Omega$. If $|x_0 - \bar{x}| < R$, then x belongs to the ball of radius R tangent to $\partial\Omega$ at $\bar{x}$.

Assume for a contradiction that

$$\mathfrak{C}_-(\Omega, \gamma) < \mathfrak{C}_-(B_R, \gamma).$$

For every $\varepsilon > 0$ such that $\mathfrak{C}_-(\Omega, \gamma) + \varepsilon < \mathfrak{C}_-(B_R, \gamma)$, there exists a set $A_\varepsilon \subset \Omega$ such that $|A_\varepsilon| \leq \mathfrak{C}_-(\Omega, \gamma) + \varepsilon$ and

$$\operatorname{essinf} v_{\Omega, A_\varepsilon, \gamma} < 0,$$

the infimum being attained at x_ε. Taking a sequence $\varepsilon \to 0$, we may assume (up to extracting suitable subsequences) that

$$1_{A_\varepsilon} \to g \text{ weakly-}\star \text{ in } L^\infty, \quad x_\varepsilon \to x_* \in \overline{\Omega}.$$

Then $\int_\Omega g = \mathfrak{C}_-(\Omega, \gamma)$. Let $v_{\Omega,g,\gamma}$ denote the solution of

$$-\Delta v = \gamma(1 - g) - (1 - \gamma)g, \, v \in H_0^1(\Omega),$$

we get

$$v_{\Omega, A_\varepsilon, \gamma} \to v_{\Omega, g, \gamma}$$

uniformly on $\overline{\Omega}$. This is a consequence of the elliptic regularity of the solutions, which are uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Consequently, $v_{\Omega,g,\gamma} \geq 0$ in Ω. Indeed, for x^* a minimum point of $v_{\Omega,g,\gamma}$ with $v_{\Omega,g,\gamma}(x^*) < 0$, we can modify g slightly to find a new function $\tilde{g}$, such that

$$0 \leq \tilde{g} \leq 1, \quad \int_\Omega \tilde{g} < \mathfrak{C}_-(\Omega, \gamma), \quad v_{\Omega, \tilde{g}, \gamma}(x^*) < 0.$$

From the density of the characteristic functions, we can find a sequence of sets $\tilde{A}_\delta$ such that $1_{\tilde{A}_\delta} \to \tilde{g}$ weakly-$\star$ in L^∞, and $|\tilde{A}_\delta| = \int_\Omega \tilde{g}$. In particular, $v_{\Omega, \tilde{A}_\delta, \gamma}(x^*) < 0$. This contradicts the definition of $\mathfrak{C}_-(\Omega, \gamma)$.

Consequently, $v_{\Omega,g,\gamma}(x^*) = 0$. There are two possibilities: either $x^* \in \Omega$, or $x^* \in \partial\Omega$. Assume first that $x^* \in \Omega$. As a consequence of the covering property, there exists a ball B of radius R such that $x_0 \in B \subset \Omega$. In particular, this implies that $v_{\Omega,g,\gamma} \geq 0$ on ∂B. The maximum principle gives

$$v_{\Omega,g,\gamma} \geq v_{B,g,\gamma}, \text{ on } B.$$

Consequently $v_{B,g,\gamma}(x^*) \leq 0$. Clearly

$$\int_B g \leq \mathfrak{C}_-(\Omega, \gamma) < \mathfrak{C}_-(B_R, \gamma). \tag{19}$$

Case 1. In case $v_{B,g,\gamma}(x^*) < 0$, we immediately get a contradiction since, as above, we can build a sequence of sets $\tilde{A}_\delta \subset B$ such that $1_{\tilde{A}_\delta} \to \tilde{g} \cdot 1_B$

weakly-$\star$ in $L^\infty(B)$, and $|\tilde{A}_\delta| = \int_B g$. By the uniform convergence, we get that $v_{B,\tilde{A}_\delta,\gamma}(x^*) < 0$, so that $\mathfrak{C}_-(B_R,\gamma) \le \int_B g$. This contradicts (19).

Case 2. In case $v_{B,g,\gamma}(x^*) = 0$, we claim that either g is itself a characteristic function, or we can find another function $\tilde{g}$ such that

$$0 \le \tilde{g} \le 1, \int_B \tilde{g} < \mathfrak{C}_-(B_R,\gamma), \text{ and } v_{B,\tilde{g},\gamma}(x^*) < 0.$$

Assume that g is a characteristic function. Then $g = 1_A$. Taking a new set $A \subset \tilde{A} \subset B$, such that $|A| < |\tilde{A}| < \mathfrak{C}_-(B_R,\gamma)$ we get by the maximum principle that $v_{B,\tilde{A},\gamma}(x^*) < 0$, in contradiction with the definition of $\mathfrak{C}_-(B_R,\gamma)$.

Assume that g is not a characteristic function on B. Then, for some value $\delta > 0$ the set $U_\delta = \{x \in B : \delta \le g(x) \le 1-\delta\}$ has positive Lebesgue measure. We put $\tilde{g} = g + s1_{U_\delta}$, where $s > 0$ is small enough such that $\int_B \tilde{g} < \mathfrak{C}_-(B_R,\gamma)$. By the maximum principle, we get $v_{B,\tilde{g},\gamma}(x^*) < 0$. In this case, we are back to Case 1.

Assume now that $x^* \in \partial\Omega$. Let $\mathbf{n}_{\mathbf{x}^*}$ be the outward normal vector at x^*. Let $\overline{x}_\varepsilon$ be the projection on $\partial\Omega$ of x_ε. Since Ω is of class C^2, we get $\overline{x}_\varepsilon \to x^*$ and that there exists a point y_ε on the segment $[\overline{x}_\varepsilon, x_\varepsilon]$ such that $\nabla v_{\Omega,A_\varepsilon,\gamma}(y_\varepsilon)\cdot \mathbf{n}_{\overline{\mathbf{x}}_\varepsilon} \ge 0$. Passing to the limit, we get

$$\nabla v_{\Omega,g,\gamma}(x^*)\cdot \mathbf{n}_{\mathbf{x}^*} \ge 0.$$

Meanwhile, x^* is a minimum point of $v_{\Omega,g,\gamma}$, so that

$$\frac{\partial v_{\Omega,g,\gamma}}{\partial n}(x^*) \le 0.$$

Hence

$$\frac{\partial v_{\Omega,g,\gamma}}{\partial n}(x^*) = 0.$$

Using the R-smoothness at x^*, the ball $B \subset \Omega$ of radius R tangent to $\partial\Omega$ at x^* stays in Ω. Since $v_{\Omega,g,\gamma} \ge 0$, by the maximum principle we get $v_{\Omega,g,\gamma} \ge v_{B,g,\gamma}$. By the Hopf maximum principle, applied to $v_{\Omega,g,\gamma} - v_{B,g,\gamma}$ on B at the minimum point $x^* \in \partial B$, we have either that

$$\frac{\partial v_{\Omega,g,\gamma}}{\partial n}(x^*) - \frac{\partial v_{B,g,\gamma}}{\partial n}(x^*) < 0,$$

or that $v_{\Omega,g,\gamma} - v_{B,g,\gamma} = 0$ on B. In the first situation,

$$\frac{\partial v_{B,g,\gamma}}{\partial n}(x^*) < 0,$$

which means that $v_{B,g,\gamma}$ takes negative values close to x^*. Then, we conclude as in Case 1, above. In the second situation, if we find a point $\overline{x} \in \partial B \cap \Omega$, we can conclude as in Case 2 since $v_{B,g,\gamma}(\overline{x}) = 0$. The alternative is that $\partial B \subset \partial\Omega$ so that $\Omega = B$, and we have a contradiction.

To prove (5) we let $m \ge 3$, and let H be an open half-space. Then

$$G_H(x,y) = c_m\Big(|x-y|^{2-m} - |x^*-y|^{2-m}\Big), \tag{20}$$

where x^* is the reflection of x with respect to ∂H, and

$$c_m = \frac{\Gamma((m-2)/2)}{4\pi^{m/2}}.$$

By (14), and monotonicity we have that

$$v_{B_R,A,\gamma}(x) \ge \gamma v_{B_R}(x) - \int_A dy\, G_{H_{\bar{x}}}(x,y), \tag{21}$$

where $H_{\bar{x}}$ is the half-space tangent to B_R at $\bar{x} \in \partial B_R$. Note that $|x^* - \bar{x}| = |\bar{x} - x|$. Moreover, $|x - y| \le |x^* - y|$, $y \in \Omega$. Hence,

$$0 \le |x-y|^{2-m} - |x^* - y|^{2-m} \le (m-2)|x - x^*||x-y|^{1-m}. \tag{22}$$

Let

$$A_x^* = \{y : |y - x| < r_A\},$$

where

$$\omega_m r_A^m = |A|. \tag{23}$$

By (20), (21), (22), and radial rearrangement of A about x, we have

$$\begin{aligned}
\int_A dy\, G_{B_R}(x,y) &\le (m-2)c_m|x - x^*| \int_A dy\, |x-y|^{1-m} \\
&\le (m-2)c_m|x - x^*| \int_{A_x^*} dy\, |x-y|^{1-m} \\
&= (m-2)mc_m\omega_m r_A|x - x^*| \\
&= 2r_A|x - \bar{x}|.
\end{aligned} \tag{24}$$

The following will be used in the proof of (5), and in the proof of (55) and (56) in Remark 6 below.

Lemma 9. *If Ω is an open set in $\mathbb{R}^m, m \ge 2$ with R-smooth boundary, and if if $\lambda(\Omega) > 0$, then*

$$v_\Omega(x) \ge \frac{|x - \bar{x}|R}{2m}. \tag{25}$$

Proof. Recall that

$$v_{B_r(c)}(x) = \frac{r^2 - |x-c|^2}{2m}.$$

We first consider the case $|x - \bar{x}| > R$. Then, by domain monotonicity of the torsion function, and (24)

$$v_\Omega(x) \ge v_{B_{|x-\bar{x}|}(x)}(x) = \frac{|x-\bar{x}|^2}{2m} \ge \frac{|x-\bar{x}|R}{2m}.$$

We next consider the case $|x - \bar{x}| \le R$. Since $\partial\Omega$ is R-smooth, there exists $B_R(c_x) \subset \Omega$ such that $|c_x - \bar{x}| = R$. Hence, by (25),

$$v_\Omega(x) \ge v_{B_R(c_x)}(x) = \frac{R^2 - |x-c_x|^2}{2m} \ge \frac{(R - |x-c_x|)R}{2m} = \frac{|x-\bar{x}|R}{2m}.$$

In either case we conclude (25). □

By (24) and (25) we have that

$$v_{B_R,A,\gamma}(x) \geq \gamma \frac{|x-\bar{x}|R}{2m} - 2r_A|x-\bar{x}|. \tag{26}$$

The right-hand side of (26) is non-negative for $r_A \leq \gamma R/(4m)$. This is, by (23), equivalent to (5).

Consider the case $m = 2$. Then

$$G_H(x,y) = \frac{1}{2\pi} \log\left(\frac{|x^*-y|}{|x-y|}\right).$$

By the triangle inequality,

$$G_{B_R}(x,y) \leq G_{H_{\bar{x}}}(x,y) \leq \frac{1}{2\pi}\log\left(\frac{|x^*-x|+|x-y|}{|x-y|}\right) \leq \frac{|x^*-x|}{2\pi|x-y|}.$$

Hence we have that

$$\int_A dy\, G_{B_R}(x,y) \leq (2\pi)^{-1}|x-x^*| \int_{A^*_{\bar{x}}} dy\, |x-y|^{-1} = 2r_A|x-\bar{x}|.$$

The remaining arguments follow those of the case $m \geq 3$, as the right-hand side above equals the right-hand side of (24).

To prove (6) we let $m \geq 3$. By scaling it suffices to prove (6) for $R = 1$. Let $a \in (0,1)$. We obtain an upper bound for a such that $v_{B_1,B_a,\gamma}(0) < 0$. Note that

$$G_{B_1}(0,y) = \frac{\Gamma((m-2)/2)}{4\pi^{m/2}}\big(|y|^{2-m} - 1\big). \tag{27}$$

Hence, by (27) we have that

$$\begin{aligned} v_{B_1,B_a,\gamma}(0) &= \gamma v_{B_1}(0) - \int_{B_a} dy\, G_{B_1}(0,y) \\ &= \frac{\gamma}{2m} - \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} m\omega_m \int_{[0,a]} dr \big(r - r^{m-1}\big) \\ &\leq \frac{\gamma}{2m} - \frac{a^2}{2m}. \end{aligned} \tag{28}$$

The right-hand side of (28) is negative for $a > \gamma^{1/2}$. This implies (6).

To prove (7) we let $m = 2$, $a \in (0,1)$, and note that

$$G_{B_1}(0,y) = -\frac{1}{2\pi}\log|y|.$$

Hence,

$$\begin{aligned} v_{B_1,B_a,\gamma}(0) &= \frac{\gamma}{4} + \int_{[0,a]} dr\, r\log r \\ &= \frac{\gamma}{4} - \frac{a^2}{4} + \frac{a^2}{4}\log a^2. \end{aligned} \tag{29}$$

Let

$$a = \left(1 + \log\left(\frac{1}{\gamma}\right)\right)^{-1/2} \gamma^{1/2}. \tag{30}$$

Then $a \in (0,1)$, and by (29) and (30),

$$v_{B_1,B_a,\gamma}(0) \le -\frac{\gamma}{4}\frac{\log\left(1+\log\left(\frac{1}{\gamma}\right)\right)}{1+\log\left(\frac{1}{\gamma}\right)} < 0.$$

This implies (7). ☐

5 Proof of Theorem 4

Proof of Theorem 4. The proof of Theorem 4 relies on some basic facts on the connection between torsion function, Green function, and heat kernel. These have been exploited elsewhere in the literature. See for example [4]. We recall that (see [11], [15], [16]) the heat equation

$$\Delta u = \frac{\partial u}{\partial t} \,\mathrm{on}\, \Omega \times \mathbb{R}^+,$$

has a unique, minimal, positive fundamental solution $p_\Omega(x, y; t)$, where $x \in \Omega$, $y \in \Omega$, $t > 0$. This solution, the heat kernel for Ω, is symmetric in x, y, strictly positive, jointly smooth in $x, y \in \Omega$ and $t > 0$, and it satisfies the semigroup property

$$p_\Omega(x, y; s + t) = \int_\Omega dz\; p_\Omega(x, z; s) p_\Omega(z, y; t),$$

for all $x, y \in \Omega$ and $t, s > 0$. If Ω is an open subset of $\mathbb{R}^m$ then, by minimality,

$$p_\Omega(x, y; t) \le p_{\mathbb{R}^m}(x, y; t) = (4\pi t)^{-m/2} e^{-|x-y|^2/(4t)},\ x \in \Omega,\ y \in \Omega,\ t > 0. \quad (31)$$

It is a standard fact that for Ω open in $\mathbb{R}^m$,

$$G_\Omega(x, y) = \int_{[0,\infty)} dt\, p_\Omega(x, y; t), \quad (32)$$

whenever the integral with respect to t converges. We have

$$v_\Omega(x) = \int_{[0,\infty)} dt \int_\Omega dy\, p_\Omega(x, y; t).$$

By the heat semigroup property, we have that for $x \in \Omega, y \in \Omega, t > 0$,

$$\begin{aligned} p_\Omega(x, y; t) &= \int_\Omega dr\, p_\Omega(x, r; t/2) p_\Omega(r, y; t/2) \\ &\le \left(\int_\Omega dr\, \big(p_\Omega(x, r; t/2)\big)^2\right)^{1/2} \left(\int_\Omega dr\, \big(p_\Omega(r, y; t/2)\big)^2\right)^{1/2} \\ &= \big(p_\Omega(x, x; t) p_\Omega(y, y; t)\big)^{1/2}. \end{aligned} \quad (33)$$

Furthermore, for all $s \in (0, t)$,

$$p_\Omega(z, z; t) \le e^{-s\lambda(\Omega)} p_\Omega(z, z; t - s). \quad (34)$$

So choosing $s = t/2$ in (34), and subsequently using (33) gives that

$$p_\Omega(x, y; t) \le e^{-t\lambda(\Omega)/3}\big(p_\Omega(x, x; t/2)p_\Omega(y, y; t/2)\big)^{1/3}p_\Omega(x, y; t)^{1/3}. \qquad (35)$$

By (31) both diagonal heat kernels in the right-hand side of (35) are bounded by $(2\pi t)^{-m/2}$, and $p_\Omega(x, y; t)^{1/3} \le (4\pi t)^{-m/6}e^{-|x-y|^2/(12t)}$. Hence by (35),

$$\begin{aligned} p_\Omega(x, y; t) &\le 2^{m/3}(4\pi t)^{-m/2}e^{-t\lambda(\Omega)/3-|x-y|^2/(12t)} \\ &\le 2^{m/3}\sup_{t>0}\big(e^{-t\lambda(\Omega)/6-|x-y|^2/(24t)}\big)e^{-t\lambda(\Omega)/6}(4\pi t)^{-m/2}e^{-|x-y|^2/(24t)} \\ &= 2^{m/3}e^{-t\lambda(\Omega)/6-|x-y|\lambda(\Omega)^{1/2}/6}(4\pi t)^{-m/2}e^{-|x-y|^2/(24t)}. \end{aligned} \qquad (36)$$

Let $c > 0$, and let r_1 be the radius of a ball of volume $\frac{c}{2}$ and $r_2 = 2^{\frac{1}{m}}r_1$ be the radius of a ball of volume c. Following the result of Lieb [20, Theorem 1], there exists a translation x of B_{r_1} such that

$$\lambda(\Omega) + \lambda(B_{r_1}) \ge \lambda(\Omega \cap B_{r_1}(x)).$$

The Kohler-Jobin inequality asserts that (see for instance [5]) there exists $C_2(m) > 0$ such that for every open set Ω,

$$\lambda(\Omega)T(\Omega)^{\frac{2}{m+2}} \ge C_2(m). \qquad (37)$$

This, together with the Lieb inequality, implies

$$T(A') = \int_{\Omega\cap B_{r_1}(x)} v_{\Omega\cap B_{r_1}(x)} \ge \left(\frac{C_2(m)}{\lambda(\Omega \cap B_{r_1}(x))}\right)^{\frac{m+2}{2}} \ge \left(\frac{C_2(m)}{\lambda(\Omega) + \lambda(B_{r_1})}\right)^{\frac{m+2}{2}}, \qquad (38)$$

where $A' = B_{r_1}(x) \cap \Omega$. We put $A = B_{r_2}(x) \cap \Omega$.

We estimate the integral of $v_{\Omega,A,\gamma}$ on the set A' as follows.

$$\begin{aligned} \int_{A'} v_{\Omega,A,\gamma} &= \int_{A'} dx\Big(\int_\Omega dy\, G_\Omega(x, y)(\gamma\mathbf{1}_{\Omega\setminus A}(y) - (1-\gamma)\mathbf{1}_A(y))\Big) \\ &= \gamma\int_{A'} dx\int_{\Omega\setminus A} dy\, G_\Omega(x, y) - (1-\gamma)\int_{A'} dx\int_A dy\, G_\Omega(x, y). \end{aligned} \qquad (39)$$

By monotonicity, we have that

$$(1-\gamma)\int_{A'} dx\int_A dy\, G_\Omega(x, y)1_A(y) \ge (1-\gamma)\int_{A'} v_{A'} = (1-\gamma)T(A'). \qquad (40)$$

For all $x \in A'$, and for all $y \in \Omega \setminus A$ we have that

$$|x - y| \ge \frac{1}{2m}\big(c/\omega_m\big)^{1/m}.$$

By (32), (36), and the preceding inequality,

$$
\begin{aligned}
&\gamma \int_{A'} dx \int_{\Omega\setminus A} dy\, G_\Omega(x,y) \\
&\le \gamma 2^{m/3} 6^{m/2} \int_{[0,\infty)} dt \int_{A'} dx \int_{\Omega\setminus A} dy\, e^{-t\lambda(\Omega)/6-|x-y|\lambda(\Omega)^{1/2}/6} p_{\mathbb{R}^m}(x,y;6t) \\
&\le \gamma 2^{m/3} 6^{m/2} e^{-(c/\omega_m)^{1/m}\lambda(\Omega)^{1/2}/(12m)} \\
&\qquad\qquad \times \int_{[0,\infty)} dt e^{-t\lambda(\Omega)/6} \int_{A'} dx \int_{\Omega\setminus A} dy\, p_{\mathbb{R}^m}(x,y;6t) \\
&\le \gamma 2^{m/3} 6^{m/2} e^{-(c/\omega_m)^{1/m}\lambda(\Omega)^{1/2}/(12m)} \\
&\qquad\qquad \times \int_{[0,\infty)} dt e^{-t\lambda(\Omega)/6} \int_{A'} dx \int_{\mathbb{R}^m} dy\, p_{\mathbb{R}^m}(x,y;6t) \\
&= \gamma 2^{m/3} 6^{(m+2)/2} e^{-(c/\omega_m)^{1/m}\lambda(\Omega)^{1/2}/(12m)} |A'| \lambda(\Omega)^{-1} \\
&\le \gamma 2^{5m/6} 3^{(m+2)/2} e^{-(c/\omega_m)^{1/m}\lambda(\Omega)^{1/2}/(12m)} c\lambda(\Omega)^{-1}. \qquad (41)
\end{aligned}
$$

By (38), (40), (39), and (41), we find

$$
\begin{aligned}
\int_{A'} v_{\Omega,A,\gamma} \le{}& \gamma 2^{5m/6} 3^{(m+2)/2} e^{-(c/\omega_m)^{1/m}\lambda(\Omega)^{1/2}/(12m)} c\lambda(\Omega)^{-1} \\
&- (1-\gamma)\Big(\frac{C_2(m)}{\lambda(\Omega)+\lambda(B_{r_1})}\Big)^{\frac{m+2}{2}}. \qquad (42)
\end{aligned}
$$

In order to bound the right-hand side of (42) from above we have

$$
\begin{aligned}
\Big(\frac{C_2(m)}{\lambda(\Omega)+\lambda(B_{r_1})}\Big)^{\frac{m+2}{2}} &\ge \Big(\frac{C_2(m)^{1/2}}{\lambda(\Omega)^{1/2}+\lambda(B_{r_1})^{1/2}}\Big)^{m+2} \\
&= \Big(\frac{c}{\omega_m}\Big)^{(m+2)/m} \Big(\frac{C_2(m)^{1/2}}{\lambda(\Omega)^{1/2}(c/\omega_m)^{1/m}+2^{1/m}\lambda(B_1)^{1/2}}\Big)^{m+2}, \qquad (43)
\end{aligned}
$$

where we have used the scaling $\lambda(B_{r_1}) = r_1^{-2}\lambda(B_1)$.

In order to bound the first term in the right-hand side of (43) from above we use the inequality $e^{-x} \le \frac{((m+2)/e)^{m+2}}{x^{m+2}}$, $x>0$. We have

$$
\begin{aligned}
&e^{-(c/\omega_m)^{1/m}\lambda(\Omega)^{1/2}/(12m)} \\
&= e^{2^{1/m}\lambda(B_1)^{1/2}/(12m)} e^{-((c/\omega_m)^{1/m}\lambda(\Omega)^{1/2}+2^{1/m}\lambda(B_1)^{1/2})/(12m)} \\
&\le e^{2^{1/m}\lambda(B_1)^{1/2}/(12m)} \\
&\qquad \times (12m(m+2)/e)^{m+2}((c/\omega_m)^{1/m}\lambda(\Omega)^{1/2}+2^{1/m}\lambda(B_1)^{1/2})^{-(m+2)}. \qquad (44)
\end{aligned}
$$

By (43) and (44) we obtain that the right-hand side of (42) is bounded from above by 0, provided

$$
c \ge C_1(m)\Big(\frac{\gamma}{1-\gamma}\Big)^{m/2}\lambda(\Omega)^{-m/2},
$$

with $C_1(m)$ given by (8). This implies the bound for $C_-(\Omega,\gamma)$ in (9).

□

6 Proof of Theorem 5

We start with the following.

Lemma 10. *There exists $\varepsilon = \varepsilon(m, \gamma) > 0$ such that for every open set $\Omega \subseteq \mathbb{R}^m$ with finite torsion and for every $x_0 \in \Omega$ the following holds*

$$\textit{if } v_\Omega(x) \le \varepsilon \textit{ for a.e. } x \in B_1(x_0) \textit{ then } v_{\Omega, B_1(x_0), \gamma}(x_0) \le 0.$$

Note that a consequence of the lemma above, for every $\delta > 0$

$$\text{if } v_\Omega(x) \le \varepsilon \text{ for a.e. } x \in B_{1+\delta}(x_0) \text{ then } v_{\Omega, B_{1+\delta}(x_0), \gamma} \le 0 \text{ on } B_\delta(x_0). \tag{45}$$

Proof. Assume for the moment that Ω is bounded and smooth. Let $x_0 \in \Omega$ such that

$$v_\Omega(x) \le \varepsilon \text{ for a.e. } x \in B_1(x_0),$$

for some value $\varepsilon > 0$ that will be specified later in the proof. We observe that $v_{\Omega, B_1(x_0), \gamma}$ is Lipschitz so that for every $r \in (0, 1)$ one can define

$$M(r) := \sup_{x \in \partial B_r(x_0)} v_{\Omega, B_1(x_0), \gamma}(x).$$

The function $M : (0, 1) \to \mathbb{R}$ is Lipschitz and bounded from above by ε. If there exists some $r \in (0, 1)$ such that $M(r) = 0$ then the assertion of the theorem is proved since one gets by the maximum principle that $v_{\Omega, B_1(x_0), \gamma} \le 0$ on $B_r(x_0)$. So we can assume that $M > 0$ on $(0, 1)$. Then, the supremum above is achieved at a point $x_r \in \partial B_r(x_0) \cap \Omega$.

Moreover,

$$M''(r) + \frac{m-1}{r} M'(r) \ge 1 - \gamma,$$

in the viscosity sense on $(0, 1)$. For every $0 < \varepsilon < R \le 1$ we introduce the equation

$$\phi''_{\varepsilon,R}(r) + \frac{m-1}{r} \phi'_{\varepsilon,R}(r) = 1 - \gamma, \text{ on } (\varepsilon, R),\ \phi_{\varepsilon,R}(\varepsilon) = M(\varepsilon), \phi_{\varepsilon,R}(R) = M(R).$$

By the comparison principle (see for instance [24, Theorem 1.1]) we get that $M \le \phi_{\varepsilon,R}$ on (R, d). In particular, this implies that ϕ is non-negative. If M is differentiable at R, then $\phi'_{\varepsilon,R}(R) \le M'(R)$.

Multiplying the equation for $\phi_{\varepsilon,R}$ by r^{m-1} and integrating between r and R gives

$$R^{m-1} \phi'_{\varepsilon,R}(R) - r^{m-1} \phi'_{\varepsilon,R}(r) = (1 - \gamma) \Big(\frac{R^m}{m} - \frac{r^m}{m} \Big).$$

Dividing by r^{m-1} and integrating over (ε, R) yields

$$R^{m-1} \phi'_{\varepsilon,R}(R) \int_\varepsilon^R \frac{1}{r^{m-1}} dr - (M(R) - M(\varepsilon)) =$$

$$= (1 - \gamma) \frac{R^m}{m} \int_\varepsilon^R \frac{1}{r^{m-1}} dr - \frac{1 - \gamma}{2m} (R^2 - \varepsilon^2).$$

Since M is Lipschitz and $\lim_{\varepsilon \to 0} \int_\varepsilon^R \frac{1}{r^{m-1}} dr = +\infty$, we get

$$\lim_{\varepsilon \to 0} \phi'_{\varepsilon,R}(R) = (1 - \gamma) \frac{R}{m}.$$

Finally,

$$M'(R) \geq (1-\gamma)\frac{R}{m}.$$

Integrating over $(0,1)$ gives

$$M(1) - M(0) \geq \frac{1-\gamma}{2m}.$$

Since $M \geq 0$,

$$M(1) \geq \frac{1-\gamma}{2m}.$$

Taking into account that $M \leq \gamma v_\Omega$, and putting

$$\varepsilon := \frac{1-\gamma}{2m\gamma},$$

concludes the proof.

Assume now that Ω is open and with finite torsion. Assume $x_0 \in \Omega$ is such that

$$v_\Omega(x) \leq \varepsilon \text{ for a.e. } x \in B_1(x_0).$$

Let $(\Omega_n)_n$ be an increasing sequence of open, smooth sets such that $\Omega = \cup_n \Omega_n$. For all n sufficiently large, $x_0 \in \Omega_n$. Moreover, by the maximum principle,

$$v_{\Omega_n}(x) \leq \varepsilon \text{ for a.e. } x \in B_1(x_0).$$

Then $v_{\Omega_n, B_1(x_0),\gamma}(x_0) \leq 0$. At the same time, $v_{\Omega_n, B_1(x_0),\gamma}$ converges to $v_{\Omega, B_1(x_0),\gamma}$ uniformly on any compact contained in $\Omega \cap B_1(x_0)$. Hence $v_{\Omega_n, B_1(x_0),\gamma}(x_0) \leq 0$. □

Proof of Theorem 5. Let Ω be open, connected, and with finite torsion. If $\mathfrak{C}_-(\Omega,\gamma) = 0$, then the inequality (10) is satisfied. Assume $\mathfrak{C}_-(\Omega,\gamma) > 0$. Then, for every $\delta > 0$ there exists $t > 0$ such that

$$\mathfrak{C}_-(t\Omega,\gamma) = (1+\delta)|B_1|. \tag{46}$$

By (4),

$$t = \Big(\frac{(1+\delta)|B_1|}{\mathfrak{C}_-(\Omega,\gamma)}\Big)^{\frac{1}{m}}. \tag{47}$$

If there exists $x_0 \in t\Omega$ such that $v_{t\Omega} \leq \varepsilon$ on $B_1(x_0)$, then by Lemma 10 we get $v_{\Omega,B_1(x_0),\gamma}(x_0) \leq 0$, so that $\mathfrak{C}_-(t\Omega,\gamma) \leq |B_1|$, in contradiction with our choice. Consequently, for every $x_0 \in t\Omega$, $\sup_{B_1(x_0)} v_{t\Omega} > \frac{1-\gamma}{2m\gamma}$. This inequality leads to a relationship between $T(t\Omega)$ and $\operatorname{diam}(t\Omega)$.

Indeed, if for some $y \in t\Omega$, $v_{t\Omega}(y) > \frac{1-\gamma}{2m\gamma}$, then for every $r > 0$

$$\int_{B_r(y)} v_{t\Omega}(x)dx \geq r^m |B_1| \Big(\frac{1-\gamma}{2m\gamma} - \frac{r^2}{2(m+2)}\Big).$$

This follows from the fact that $x \mapsto v_{t\Omega}(x) + \frac{|x-y|^2}{2m}$ is subharmonic on $\mathbb{R}^m$. We have extended v_Ω to all of $\mathbb{R}^m$ by putting $v_\Omega(x) = 0$ on $\mathbb{R}^m \setminus \Omega$.

Choosing r such that

$$\frac{r^2}{2(m+2)} = \frac{1-\gamma}{4m\gamma}, \tag{48}$$

we get

$$\int_{B_r(y)} v_{t\Omega}(x)dx \geq \frac{(m+2)^{m/2}}{2^{(m+4)/2}m^{(m+2)/2}}\Big(\frac{1-\gamma}{\gamma}\Big)^{(m+2)/2}|B_1|.$$

Assume N is an integer such that

$$N(2r+2) \leq \mathrm{diam}(t\Omega) \leq (N+1)(2r+2).$$

Then,

$$T(t\Omega) \geq N\frac{(m+2)^{\frac{m}{2}}}{2^{(m+4)/2}m^{(m+2)/2}}\Big(\frac{1-\gamma}{\gamma}\Big)^{\frac{m+2}{2}}|B_1|.$$

If $N \geq 1$, then using the inequality $N+1 \leq 2N$ we get

$$\mathrm{diam}(t\Omega) \leq 2(2r+2)\Big(\frac{(m+2)^{m/2}}{2^{(m+4)/2}m^{(m+2)/2}}\Big(\frac{1-\gamma}{\gamma}\Big)^{(m+2)/2}|B_1|\Big)^{-1}T(t\Omega) \quad (49)$$

If $N = 0$, then we observe that $\mathrm{diam}(t\Omega) \leq 2r+2$. Inequality (9) (which follows from Theorem 4) gives

$$\mathfrak{C}_{-}(t\Omega,\gamma) \leq C_1(m)C_2(m)^{-m/2}\left(\frac{\gamma}{1-\gamma}\right)^{m/2}T(t\Omega)^{m/(m+2)}.$$

By (46),

$$(1+\delta)|B_1| \leq C_1(m)C_2(m)^{-m/2}\left(\frac{\gamma}{1-\gamma}\right)^{m/2}T(t\Omega)^{m/(m+2)}.$$

Finally,

$$\mathrm{diam}(t\Omega) \leq 2r+2 \leq (2r+2)\Big(\frac{C_1(m)C_2(m)^{-m/2}}{(1+\delta)|B_1|}\Big)^{\frac{m+2}{m}}\left(\frac{\gamma}{1-\gamma}\right)^{(m+2)/2}T(t\Omega). \quad (50)$$

We observe that the γ-dependence in both (49) and (50) is the same. Taking the larger of the two m-dependant constants which show up in front of $T(t\Omega)$ in (49) and (50), replacing t from (47), and letting $\delta \to 0$, and using (48) concludes the proof. □

7 Proof of Theorem 6

The proof of Theorem 6 requires the extension of the constant $\mathfrak{C}_{-}(\Omega,\gamma)$ to quasi-open sets. A proper introduction to the Laplace equation on quasi-open sets, capacity theory, and gamma convergence can be found in [17] and [7]. We prefer, for expository reasons, to avoid an extensive introduction to this topic, and refer the interested reader to [7, Sections 4.1 and 4.3] where all terminology used below can be found.

The key observation is that the class of quasi-open sets is the largest class of sets where the Dirichlet-Laplacian problem is well defined in the Sobolev space H^1_0, and satisfies a strong maximum principle (see [13]). Of course any open set is also quasi-open. Although the reader may only be interested in open sets, we are forced to work with quasi-open ones since the crucial step of the proof is the existence of a quasi-open set Ω^* which maximises the left-hand side of (11).

The strategy of the proof is as follows. We analyse the shape optimisation problem

$$\sup\left\{\frac{\mathfrak{C}_-(\Omega,\gamma)}{|\Omega|} : \Omega \subseteq \mathbb{R}^m,\ \Omega \text{ quasi-open with } 0<|\Omega|<\infty\right\}, \tag{51}$$

and prove in Step 1 below the existence of a maximiser Ω^*. Denoting $C'(m,\gamma) = \mathfrak{C}_-(\Omega^*,\gamma)/|\Omega*|$ we then prove in Step 2 that $C'(m,\gamma) < \gamma$ by a direct estimate on Ω^*.

We start with the following observation. Assume that $(\Omega_n)_n$ is a sequence of quasi-open sets of $\mathbb{R}^m$, $|\Omega_n| \le 1$, such that v_{Ω_n} converges strongly in $L^2(\mathbb{R}^m)$, and pointwise a.e. to some function v. Let us denote $\Omega := \{v>0\}$. We then have

$$\mathfrak{C}_-(\Omega,\gamma) \ge \limsup_{n\to+\infty} \mathfrak{C}_-(\Omega_n,\gamma). \tag{52}$$

Indeed, in order to prove this assertion let us consider a set $A \subseteq \Omega$ such that $\operatorname{essinf} v_{\Omega,A,\gamma} < 0$. We have

$$1_\Omega(x) \le \liminf_{n\to+\infty} 1_{\Omega_n}(x) \text{ a.e. } x \in D, \tag{53}$$

and hence

$$1_{\Omega_n\cap A} \to 1_A \text{ in } L^1(\mathbb{R}^m).$$

Following [7, Lemma 4.3.15], there exists larger sets $\tilde{\Omega}_n \supset \Omega_n$, $|\tilde{\Omega}_n| \le 2$, such that for a subsequence (still denoted with the same index)

$$\lim_{n\to+\infty} v_{\tilde{\Omega}_n,A\cap\Omega_n,\gamma}(x) = v_{\Omega,A,\gamma}(x), \text{ for a.e. } x \in \mathbb{R}^m.$$

Since $\operatorname{essinf} v_{\Omega,A,\gamma} < 0$, we get for n large enough that $\operatorname{essinf} v_{\tilde{\Omega}_n,A\cap\Omega_n\,\gamma} < 0$ for n large enough. Lemma 8 (which also holds in the class of quasi-open sets) implies that $\operatorname{essinf} v_{\Omega_n,A\cap\Omega_n,\gamma} < 0$, since the right-hand side equals to γ, $\gamma > 0$ on $\tilde{\Omega}_n \setminus \Omega_n$. Consequently, $\mathfrak{C}_-(\Omega_n,\gamma) \le |\Omega_n \cap A|$. Passing to the limit,

$$\limsup_{n\to+\infty} \mathfrak{C}_-(\Omega_n,\gamma) \le |A|,$$

which implies the assertion.

Let us prove now that the shape optimisation problem (51) has a solution. In order to prove this result, it is enough to consider a maximising sequence (Ω_n) of quasi-open, quasi-connected subsets of $\mathbb{R}^m$, with $|\Omega_n| = 1$. We first notice that the diameters of Ω_n are uniformly bounded, so that up to a translation all of them are subsets of the same ball B. This is a consequence of Theorem 5 which by approximation holds as well on quasi-open, quasi-connected sets. Indeed, this is essentially a consequence of (45) which passes to the limit by approximation.

Then, the existence result is immediate from the compact embedding of $H^1_0(B) \rightharpoonup L^2(B)$ and the observation above: there exists a subsequence such that v_{Ω_n} converges strongly in $L^2(\mathbb{R}^m)$ and pointwise a.e. to some function v. Taking $\Omega^* := \{v>0\}$, and using the upper semi-continuity result (52) together with the lower semicontinuity of the Lebesgue measures coming from (53), we conclude that Ω^* is optimal.

8 Proof of Theorem 7, and further remarks

Proof of Theorem 7. For a measurable set $A \subset \Omega$, we denote

$$m(A) := \text{essinf } v_{\Omega,A,\gamma}.$$

Note that the smoothness of $\partial\Omega$ implies that $v_{\Omega,A,\gamma} \in C^{1,\alpha}(\overline{\Omega})$.

First we extend the shape functional m, on the closure of the convex hull of

$$\{\gamma 1_{\Omega\setminus A} - (1-\gamma)1_A : A \subset \Omega, |A| = c\}.$$

Denote by

$$\mathcal{F} := \{f \in L^\infty(\Omega) : -(1-\gamma) \le f \le \gamma, \int_\Omega f = \gamma|\Omega| - c\}.$$

One naturally extends the functional m to the set $\mathcal{F}$ by defining $v_{\Omega,f,\gamma}$ as the solution of $-\Delta v = f$ in $H_0^1(\Omega)$. We shall prove in the sequel that the relaxation of the shape optimisation problem (12) on the set $\mathcal{F}$ has a solution in $\mathcal{F}$. Precisely, we solve

$$\min\{m(f) : f \in \mathcal{F}\}. \tag{54}$$

Clearly, $\mathcal{F}$ is compact for the weak-$\star$ L^∞-topology, so that we can assume that $(f_n)_n$ is a minimising sequence which converges in weak-$\star$ L^∞ to f. We know, by the Calderon-Zygmund inequality, that $\left(v_{\Omega,f_n,\gamma}\right)_n$ are uniformly bounded in $W^{2,p}(\Omega)$, for every $p < \infty$. In particular, for p large enough, this implies that $v_{\Omega,f_n,\gamma}$ converges uniformly to $v_{\Omega,f,\gamma}$. Consequently, this implies that $m(f_n)$ converges to $m(f)$ so that f is a solution to the optimisation problem (54).

Secondly we prove there exists some set A such that $f = \gamma 1_{\Omega\setminus A} - (1-\gamma)1_A$. To prove this we exploit both the concavity property of the map $f \mapsto m(f)$, and the structure of the PDE. Assume for contradiction that the set

$$A_\varepsilon := \{x \in \Omega, -(1-\gamma) + \varepsilon \le f(x) \le \gamma - \varepsilon\}$$

has non-zero measure, for some $\varepsilon > 0$. Let $A_1, A_2 \subset A_\varepsilon$ be two disjoint sets, such that $|A_1| = |A_2|$. We consider the functions $f_1 = f + t1_{A_1} - t1_{A_2}$, and $f_2 = f - t1_{A_1} + t1_{A_2}$, for $t \in (-\varepsilon, \varepsilon)$. Then, $f_1, f_2 \in \mathcal{F}$, and by linearity we have

$$v_{\Omega,f,\gamma} = \frac{1}{2}v_{\Omega,f_1,\gamma} + \frac{1}{2}v_{\Omega,f_2,\gamma}.$$

Consequently,

$$m(f) \ge \frac{1}{2}m(f_1) + \frac{1}{2}m(f_2),$$

with strict inequality if the point x^* where $v_{\Omega,f,\gamma}$ is minimised also minimises $v_{\Omega,f_1,\gamma}$ and $v_{\Omega,f_2,\gamma}$. Moreover, we have $v_{\Omega,f,\gamma}(x_*) = v_{\Omega,f_1,\gamma}(x_*) = v_{\Omega,f_2,\gamma}(x_*)$. We distinguish between two situations: $v_{\Omega,f,\gamma}(x_*) = 0$, and $v_{\Omega,f,\gamma}(x_*) < 0$. If we are in the first situation, then x^* could belong to $\partial\Omega$. In this case, for all admissible sets A we have $v_{\Omega,A,\gamma} \ge 0$, the minimal value, which is 0 being attained on $\partial\Omega$. In this case, every admissible set A is a solution to the shape optimisation problem.

If we are in the second situation, then necessarily $x^* \in \Omega$. By linearity, from $v_{\Omega,f,\gamma}(x_*) = v_{\Omega,f_1,\gamma}(x_*)$ we get

$$v_{\Omega,1_{A_1},0}(x_*) = v_{\Omega,1_{A_2},0}(x_*).$$

In particular, for every pair of points $x, y \in A_\varepsilon \setminus \{x^*\}$ with density 1 in A_ε we get

$$G_\Omega(x^*, x) = G_\Omega(x^*, y).$$

Since G_Ω is harmonic on $\Omega \setminus \{x^*\}$, we get that G_Ω is constant in $\Omega \setminus \{x^*\}$, in contradiction with the fact that it is a fundamental solution.

Finally, this implies that $|A_\varepsilon| = 0$ for every $\varepsilon > 0$. Hence f is a characteristic function.

Remark 1. Clearly, the solution of the shape optimisation problem above is, in general, not unique. If the minimal value is 0, then any admissible set A is a solution. If the minimal value is strictly negative, then there are geometries with non uniqueness. For example if Ω is the union of two disjoint balls with the same radius. Then A is a subset of one of the two balls.

Remark 2. Assume $\Omega = B_R$, and $|B_R| \geq c \geq \gamma |B_R|$. The solution to the shape optimisation problem (12) is given by the (concentric) ball B_{r_c}, of mass c, $c = |B_{r_c}|$. Indeed, there are two possibilities. In the case that $A \subset B_R$ has measure c and $v_{B_R,A,\gamma} \leq 0$, we can use directly Talenti's theorem to conclude (applied to $-v_{B_R,A,\gamma}$).

Assume now that $v_{B_R,A,\gamma}$ changes sign on B_R. We define the sets $\Omega^+ = \{v_{B_R,A,\gamma} > 0\}$ and $\Omega^- = \{v_{B_R,A,\gamma} < 0\}$. In view of Theorem 1, we have that $|A \cap \Omega^+| \leq \gamma |\Omega^+|$ and $|A \cap \Omega^-| \geq \gamma |\Omega^-|$. We use Talenti's theorem on Ω^-, and get that the essential infimum of the function $v_{B_{R'},B_{r'},\gamma}$ is not larger than the infimum of $v_{B_R,A,\gamma}$, where $B_{R'}, B_{r'}$ are the balls centred at the origin of measures $|\Omega^-|$, $|\Omega^- \cap A|$, respectively. We claim that $v_{B_R,B_{r_c},\gamma} \leq v_{B_{R'},B_{r'},\gamma}$. Indeed, making a suitable rescaling by a factor $t \geq 1$ such that $|t(\Omega^- \cap A)| = c \geq \gamma |B_R|$, the function $v_{tB_{R'},tB_{r'},\gamma}$ has an essential infimum lower than that of $v_{B_{R'},B_{r'},\gamma}$. We finally notice that $v_{B_R,B_{r_c},\gamma} \leq v_{tB_{R'},B_{r_c},\gamma}$. Indeed, this is a consequence of the fact that $v_{tB_{R'},B_{r_c},\gamma}$ is equal to $\min\{-\delta, v_{B_R,B_{r_c},\gamma}\} + \delta$, for a suitable $\delta > 0$.

Remark 3. Assume $\Omega = B_R$. Let $0 < c < |B_R|$ and denote B_{r_c} the ball with the same center as B_R and of volume c. For every radial set A of volume c we have

$$v_{B_R,B_{r_c},\gamma} \leq v_{B_R,A,\gamma}.$$

Indeed, let us denote for simplicity $v = v_{B_R,A,\gamma}$ and $v_c = v_{B_R,B_{r_c},\gamma}$. Using the fact that both v and v_c are radial, we get

$$\begin{aligned}
-r^{m-1}v'(r) &= \int_0^r s^{m-1}(\gamma 1_{B_R \setminus A} - (1-\gamma)1_A)ds \\
&= \frac{1}{\omega_{m-1}} \int_{B_r} \big(\gamma 1_{B_R \setminus A} - (1-\gamma)1_A\big),
\end{aligned}$$

$$\begin{aligned}
-r^{m-1}v_c'(r) &= \int_0^r s^{m-1}(\gamma 1_{B_R \setminus B_{r_c}} - (1-\gamma)1_{B_{r_c}})ds \\
&= \frac{1}{\omega_{m-1}} \int_{B_r} \big(\gamma 1_{B_R \setminus B_{r_c}} - (1-\gamma)1_{B_{r_c}}\big),
\end{aligned}$$

where, for a radial set $E \subseteq B_R$, we define (with abuse of notation), $1_E(r)$ being the value of 1_E on the sphere of radius r.

Since for all $r \in (0, R)$,

$$\int_{B_r} \left(\gamma 1_{B_R \setminus A} - (1-\gamma) 1_A\right) \geq \int_{B_r} \left(\gamma 1_{B_R \setminus B_{r_c}} - (1-\gamma) 1_{B_{r_c}}\right),$$

we get that for all $r \in (0, R)$

$$-r^{m-1} v'(r) \geq -r^{m-1} v_c'(r).$$

Hence

$$\int_r^1 v_c'(s) ds \geq \int_r^1 v'(s) ds,$$

and

$$-v_c(r) \geq -v(r).$$

This concludes the proof. Moreover, the infimum value of v_c is attained either at 0 or at R, as v_c' is positive on some interval $(0, \alpha)$ and negative on (α, R).

For $\gamma = \frac{1}{2}$, we can compute the value of c such that $v_c(0) = v_c(R) = 0$. Indeed, in $\mathbb{R}^2$, the corresponding value r_c is the solution of

$$\frac{r^2}{2} - \frac{1}{4} - r^2 \ln r = 0.$$

An estimate of the solution is $r_c \approx 0.432067$.

Remark 4. Assume $\Omega = B_R$, and $\mathfrak{C}_-(B_R, \gamma) < c < \gamma |B_R|$. The solution of the shape optimisation problem is non-trivial in this case. While we do not know the general solution, we can observe a symmetry breaking phenomenon: the solution is not radially symmetric for small values of c.

Let $\gamma = \frac{1}{2}, r_c = 0.432$ just below the value computed in the previous remark. Then, for every radial set A, the essential infimum of $v_{B_R, A, \frac{1}{2}}$ is equal to 0. Meanwhile, there exists a non-radial set A which gives a lower essential infimum. This fact is observed numerically, if for instance the set A is a disc, centred at $(0.52, 0)$ of radius $r_c = 0.432$. Of course, the fact that in this case the essential infimum is strictly negative can be directly deduced from estimates of the Poisson formula. In Figures 1 and 2 below, we display the (rescaled) numerical solutions computed with MATLAB.

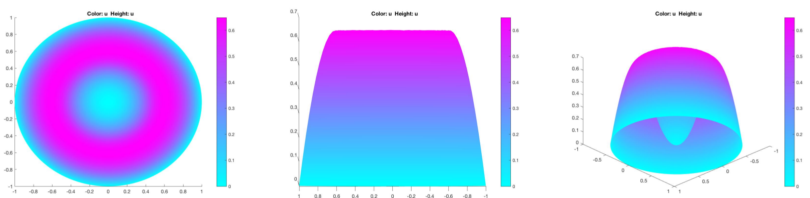

Figure 1: Negative mass displayed in the disc centred at 0 of radius $r = 0.432$: the essential infimum is 0.

If c is less than the critical value, the infimum is equal to 0, and is attained for an infinite number of solutions to the shape optimisation problem.

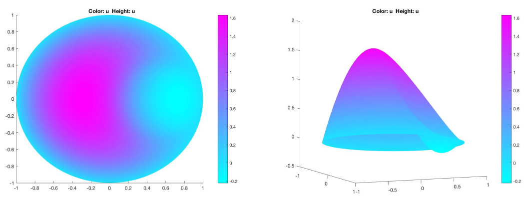

Figure 2: Negative mass placed on the disc centred at $(0.52, 0)$ of radius $r = 0.432$: the essential infimum is negative.

Remark 5. The solutions of the following shape optimisation problems

$$\max\Big\{\int_\Omega v_{\Omega,A,\gamma} : A\subset\Omega, |A| = c\Big\},$$

and

$$\min\Big\{\int_\Omega v_{\Omega,A,\gamma} : A\subset\Omega, |A| = c\Big\}.$$

are immediate. Indeed, we observe that

$$\int_\Omega v_{\Omega,A,\gamma} = \gamma\int_\Omega v_\Omega - \int_A v_\Omega.$$

Hence, the position of the set A is a suitable lower/upper level set of v_Ω.

Remark 6. If $|A| \le \mathfrak{C}_-(B_R,\gamma)$ then $v_{\Omega,A,\gamma} \ge 0$, and $\int_\Omega v_{\Omega,A,\gamma} \ge 0$. Below we improve the bound $|A| \le \left(\frac{\gamma}{4m}\right)^m \omega_m R^m$ in (5) for $\int_\Omega v_{\Omega,A,\gamma} \ge 0$ to hold.

Let $\Omega \subset \mathbb{R}^m, m \ge 2$, be an open set with finite measure and a C^2, R-smooth boundary. Let $\gamma > 0$ and let $v_{\Omega,A,\gamma}$ be the solution of (3). If either $m \ge 3$, and

$$|A| \le \frac{m}{6(m-1)^2}\gamma\omega_m R^m, \tag{55}$$

or $m = 2$, and

$$|A| \le \frac{10+7\sqrt{7}}{324}\gamma\pi R^2, \tag{56}$$

then $\int_\Omega v_{\Omega,A,\gamma} \ge 0$.

Proof. First consider the case $m \ge 3$. By Lemma 9 and the coarea formula, we have for $a > 0$ that,

$$\begin{aligned}\int_\Omega v_\Omega &\ge \int_{\{x\in\Omega:|x-\bar{x}|<a\}} dx\,\frac{|x-\bar{x}|R}{2m}\\ &\ge \int_{[0,a]} d\theta\,\frac{R\theta}{2m}\mathcal{H}^{m-1}(\partial\Omega_\theta),\end{aligned} \tag{57}$$

where $\mathcal{H}^{m-1}(\partial\Omega_\theta)$ denotes the $(m-1)$-dimensional Hausdorff measure of the parallel set $\{x \in \Omega : |x-\bar{x}| = \theta.\}$ It was shown in Lemma 5 in [2] that for an open, bounded set Ω with a C^2, R-smooth boundary,

$$\mathcal{H}^{m-1}(\partial\Omega_\theta) \ge \left(1 - \frac{(m-1)\theta}{R}\right)\mathcal{H}^{m-1}(\partial\Omega),\ \theta \ge 0. \tag{58}$$

By (57) and (58) we obtain

$$\int_\Omega v_\Omega \geq \frac{R}{2m}\left(\frac{a^2}{2} - \frac{(m-1)a^3}{3R}\right)\mathcal{H}^{m-1}(\partial\Omega).$$

Optimising over a yields,

$$\int_\Omega v_\Omega \geq \frac{R^3}{12m(m-1)^2}\mathcal{H}^{m-1}(\partial\Omega). \tag{59}$$

By the isoperimetric inequality we have

$$\mathcal{H}^{m-1}(\partial\Omega) \geq m\omega_m^{1/m}|\Omega|^{(m-1)/m}. \tag{60}$$

Since Ω contains a ball of radius R, $|\Omega| \geq \omega_m R^m$. Hence, by (59) and (60),

$$\mathcal{H}^{m-1}(\partial\Omega) \geq m\omega_m^{(m-2)/m}R^{m-3}|\Omega|^{2/m}.$$

This, together with (59) yields

$$\int_\Omega v_\Omega \geq \frac{\omega_m^{(m-2)/m}}{12(m-1)^2}R^m|\Omega|^{2/m}. \tag{61}$$

By Talenti's theorem,

$$\begin{aligned}\int_A v_\Omega &\leq \int_{A^*} v_{\Omega^*}\\ &= 2^{-1}\omega_m\int_{[0,r_A]} dr\big(R_\Omega^2 - r^2\big)r^{m-1}\\ &\leq (2m)^{-1}\omega_m R_\Omega^2 r_A^m\\ &= (2m)^{-1}\omega_m^{-2/m}|\Omega|^{2/m}|A|.\end{aligned} \tag{62}$$

By (3), (61), and (62) we have that

$$\begin{aligned}\int_\Omega v_{\Omega,A,\gamma} &= \gamma\int_\Omega v_\Omega - \int_A v_\Omega\\ &\geq \frac{\omega_m^{(m-2)/m}}{12(m-1)^2}\gamma R^m|\Omega|^{2/m} - (2m)^{-1}\omega_m^{-2/m}|\Omega|^{2/m}|A|.\end{aligned} \tag{63}$$

This implies that $\int_\Omega v_{\Omega,A,\gamma} \geq 0$ for all measurable $A \subset \Omega$ satisfying (55).

Next consider the planar case. By Lemma 9, we have for any $\alpha \in (0,1)$,

$$\int_{\{x\in\Omega:|x-\bar{x}|\geq\alpha R\}} v_\Omega \geq \frac{\alpha R^2}{4}|\{x \in \Omega : |x-\bar{x}| \geq \alpha R\}|. \tag{64}$$

By the coarea formula, Lemma 9, and (58), we find

$$\int_{\{x\in\Omega:|x-\bar{x}|<\alpha R\}} v_\Omega \geq \frac{R^3}{4}\Big(\frac{\alpha^2}{2} - \frac{\alpha^3}{3}\Big)\mathcal{H}^1(\partial\Omega). \tag{65}$$

By Lemma 5 in [2],

$$\mathcal{H}^1(\partial\Omega_\theta) \leq \frac{R}{R-\theta}\mathcal{H}^1(\partial\Omega),\ 0 \leq \theta < R. \tag{66}$$

By the coarea formula, and (66), we find

$$
\begin{aligned}
|\{x \in \Omega : |x-\bar{x}| \le \alpha R\}| &\le \mathcal{H}^1(\partial\Omega) \int_{[0,\alpha R]} d\theta \, \frac{R}{R-\theta} \\
&\le \alpha(1-\alpha)^{-1}\mathcal{H}^1(\partial\Omega)R. \qquad (67)
\end{aligned}
$$

Putting (64), (65), and (67) together gives that

$$
\begin{aligned}
\int_\Omega v_\Omega &\ge \frac{\alpha R^2}{4}|\{x \in \Omega : |x-\bar{x}| \ge \alpha R\}| \\
&\quad + \frac{1}{24}\alpha(1-\alpha)(3-2\alpha)R^2|\{x \in \Omega : |x-\bar{x}| \le \alpha R\}| \\
&\ge \min\Big\{\frac{\alpha}{4}, \frac{1}{24}\alpha(1-\alpha)(3-2\alpha)\Big\}R^2|\Omega| \\
&= \frac{1}{24}\alpha(1-\alpha)(3-2\alpha)R^2|\Omega|.
\end{aligned}
$$

We choose $\alpha = \frac{1}{6}(5-\sqrt{7})$ so as to maximise the above right-hand side, and obtain

$$
\int_\Omega v_\Omega \ge \frac{10+7\sqrt{7}}{1296}R^2|\Omega|. \qquad (68)
$$

Formula (62) for $m=2$, (68), and the first equality in (63) yield,

$$
\int_\Omega v_{\Omega,A,\gamma} \ge \Big(\frac{10+7\sqrt{7}}{1296}\gamma R^2 - \frac{1}{4\pi}|A|\Big)|\Omega|.
$$

The above right-hand side is non-negative for all measurable $A \subset \Omega$ satisfying (56). □

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